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Limit Clusters in the Inviscid Burgers Turbulence with Certain Random Initial Velocities

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We study the infinite time shock limits given certain Markovian initial velocities to the inviscid Burgers turbulence. Specifically, we consider the one-sided case where initial velocities are zero on the negative half-line and follow a time-homogeneous nice Markov process X on the positive half-line. Finite shock limits occur if the Markov process is transient tending to infinity. They form a Poisson point process if X is spectrally negative. We give an explicit description when X is furthermore spatially homogeneous (a Lévy process) or a self-similar process on $(0, \infty)$. We also consider the two-sided case where we suppose an independent dual process in the negative spatial direction. Both spatial homogeneity and an exponential Lévy condition lead to stationary shock limits.

KEY WORDS: Inviscid Burgers equation; random initial velocity; shock structure; Markov processes; self-similar processes; spectrally negative Lévy processes.

1. INTRODUCTION

The viscid Burgers equation

$$\partial_t u^{(\varepsilon)} + u^{(\varepsilon)}\,\partial_x u^{(\varepsilon)} = \varepsilon \partial_{xx}^2 u^{(\varepsilon)}, \qquad u^{(\varepsilon)}(0, x) = u_0(x), \quad x \in \mathbb{R}$$

is known to describe in the inviscid limit $\varepsilon \downarrow 0$ the velocity field $u(t, x) = \lim_{\varepsilon \downarrow 0} u^{(\varepsilon)}(t, x)$, $t \geqslant 0$, $x \in \mathbb{R}$, of an infinitesimal particle system that performs completely inelastic collisions. Although it was introduced by Burgers as a model for the turbulence of compressible fluids typically

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supposed to have a small viscosity $\varepsilon > 0$, applications in many other fields have come up since, e.g., cluster formation in the universe or chemical interface growth. See Burgers,[1] Woyczynski[2] and their references for discussions of these and other physical motivations, Hopf[3] for the mathematical solution in the viscid and inviscid case, the latter leading to a geometrical analysis which we recall below (Section 2).

This model is of particular interest when allowing random initial data, i.e., when u_0 is a stochastic process indexed by the spatial variable $x \in \mathbb{R}$. Many different classes of processes have been considered, e.g., white noise,[1, 4–7] Brownian motion,[8–10] Lévy processes[11, 10] and fractional Brownian motion.[12] Due to Lax' entropy condition[13] $u(\cdot, t)$ has no positive jumps at positive times $t > 0$. We shall here consider a large class of Markov processes with no positive jumps as initial velocities and then specialize on Lévy processes and self-similar processes of order $\alpha \geqslant 0$.

Quantities of interest are the velocities and the shock structure of the model at positive times, i.e., the clusters that are formed due to the inelastic collisions and the so-called regular particles that have not participated in shocks. In this paper we study, in a sense, the convergence of shock structure and cluster velocities as time tends to infinity. Tribe and Zaboronski[14] considered compactly supported white noise initial velocities and showed (indeed for large classes of compactly supported initial data) that the shock structure degenerates to two large clusters pushing the margins. One is then naturally led to estimating mass and velocity behaviour of these two clusters. We shall encounter a completely different behaviour.

We start with a one-sided situation, i.e.,

$$u_0(x) = 0 \qquad \text{for} \quad x < 0 \qquad \text{and} \qquad u_0(x) = X_x \qquad \text{for} \quad x \geqslant 0$$

where X is a càdlàg Markov process. An important observation is that one can calculate the infinite time limit from Bertoin[10] where X is more specifically a spectrally negative Lévy process. Furthermore the methods can be refined to larger classes of Markov processes. We focus on processes satisfying the assumption $X_x \to \infty$, $x \to \infty$. This assumption ensures that even in the limit as t tends to infinity, there are no infinite limit clusters; this follows from the fact that the speed of a given particle is bounded above by the maximal speed attained to its left and below by the minimal speed attained to its right. In the limit it is heuristically obvious that clusters and regular particles (if any) are ordered by their velocities, i.e., a limit cluster made from particles started from $[a_1, a_2)$ is slower than and a cluster $[a_3, a_4)$, $a_2 < a_3$. Note however, that their velocities are positive (apart from an initial interval from which particles may move to the left). Thus, one cannot associate a limit location. Our main result says, that if X

has no positive jumps, then the limiting shock structure can be described in terms of a Poisson point process on $[x_0, \infty) \times (0, \infty)$ whose points (u, d) correspond to clusters of total size d and common speed u. In the case of a Lévy process or a self-similar Markov process we make the intensity measure explicit and derive formulas for the law of the limit cluster size of a given particle etc. For the corresponding two-sided analogue

$$u_0(-x) = \tilde{X}_{x-} \qquad \text{and} \qquad u_0(x) = X_x \qquad \text{for} \quad x \geqslant 0$$

for two independent identically distributed Lévy processes X and $-\tilde{X}$ (or their exponential process), we show that the process of limit cluster sizes $\ell(y)$ of particles started from $y \in \mathbb{R}$, is stationary.

Note that our limits are unscaled limits. Hence, they provide a complement to the wide literature on scaling limits in Burgers turbulence, cf. Leonenko,[15] Woyczynski[2] and their references.

The rest of the paper is organised as follows. Section 2 deals with the analysis of limit clusters in a deterministic setting. In Section 3 we treat the random setting, successively for one-sided Markov processes, one-sided Lévy processes, self-similar Markov processes and the two-sided case.

2. GEOMETRICAL DESCRIPTION OF LIMIT CLUSTERS

The inviscid Burgers turbulence describes an infinitesimal particle system whose particles are initially uniformly distributed on $\mathbb{R}$ (according to Lebesgue measure) and perform inelastic collisions, i.e., two particles (particle clumps) that meet form a larger particle clump under conservation of masses and momenta (loss of energy). As time $t > 0$ evolves, the shocks generally lead to a mixture of particle clumps, infinitesimal particles (their positions are called regular points) and empty areas (called rarefaction intervals). A description in terms of the initial velocities u_0 can be done based on the so-called inverse Lagrangian

$$a_t(x) = \arg^+ \min_{a \in \mathbb{R}} \left\{ \int_0^a (u_0(y) - (x-y)/t)\, dy \right\}$$

where $\arg^+ \min_{a \in \mathbb{R}}$ assigns the right-most value a for which the expression is minimal. For quite general u_0, e.g., u_0 càdlàg with

$$\liminf_{|y| \to \infty} y^{-1} u_0(y) \geqslant 0$$

a_t is increasing and right-continuous, and describes the shock structure at time t as follows.

(i) An interval $I_g=[g,d)$ where a_t is constant is a rarefaction interval, i.e., at time t there are no particles in I_g.

(ii) A continuity location x of a_t outside the union of I_g, $g\geqslant 0$, contains a regular particle, i.e., an infinitesimal particle which has not been involved in any shocks. It started from $a_t(x)$ and has kept its initial speed up to time t.

(iii) A jump location x of a_t contains a cluster whose size at time t is equal to the jump size $s=\Delta a(x)$. This cluster is built from the particles with initial positions in $[a(x-), a(x))$. If the minimum defining $a_t(x)$ is attained more than twice, there is a shock at time t which merges particle clumps and/or infinitesimal particles as described by the minimum positions: each two neighbouring minimum positions correspond to a clump whose size was the distance of the two positions, and if the minimum is attained on an interval, the corresponding particles involved were infinitesimal.

This description is well-known. Cf. e.g., Hopf,[3] also for the link to Burgers equation. We also refer to ref. 16 where we gave an elementary derivation.

Note that this analysis can be looked at in a much more geometrical way as follows: consider the graph of the potential function $\psi(a)=\int_0^a u_0(y)\,dy$. Then $a_t(x)$ is the right-most touching point of the maximal parabola $q_c(a)=c-(a-x)^2/2t$ minorizing ψ. Hence the function a_t is defined by the family of maximal minorizing parabolas of curvature $-1/2t$. It is natural to suppose that minorizing parabolas of curvature 0, i.e., straight lines, describe the limit state as t tends to infinity. This is indeed the case. We shall now give an analogous description of the limit clusters as t tends to infinity.

Proposition 1. Consider an inviscid Burgers turbulence model with càdlàg initial velocities u_0. If for all $y\in\mathbb{R}$

$$-\infty\leqslant U^-:=\limsup_{z\to-\infty} u_0(z)\leqslant u_0(y)\leqslant \liminf_{z\to\infty} u_0(z)=:U^+\leqslant\infty \tag{1}$$

the function

$$a(x)=\arg^+\min_{a\in\mathbb{R}}\left\{\int_0^a (u_0(y)-x)\,dy\right\},\qquad x\in(U_-,U^+)$$

is well-defined, right-continuous and increases from $Y^-:=\inf\{y\in\mathbb{R}: u_0(y)>U^-\}\geqslant-\infty$ to $Y^+:=\sup\{y\in\mathbb{R}: u_0(y)<U^+\}\leqslant\infty$. a describes the limit structure as follows.

(i) An interval $I_g=[g, d)$ where a is constant consists of speeds $g \leqslant x < d$ that are not attained in the limit.

(ii) A continuity point x of a outside the union of I_g, $g \geqslant 0$, is the speed of exactly one regular particle.

(iii) A jump location x of a is the speed of limit clusters of total size $s=\Delta a(x)$. They are built from the particles with initial locations in $[a(x-), a(x))$. In fact, there is only one cluster except when the minimum defining $a(x)$ is attained more than twice. In this case, the interval between each two neighbouring minimum positions forms one such cluster and if the minimum is attained on an interval, the corresponding particles are regular.

(iv) The limit speeds U^- and U^+ are maintained by the regular particles from $(-\infty, Y^-)$ and $[Y^+, \infty)$, respectively.

In our applications we shall always have $U^+=\infty$ (and $Y^+=\infty$). However, choosing $U^-=0$ and $Y^-=0$ corresponds to one-sided initial conditions. Some of our situations later correspond to a slightly more general setting where the lower bound U^- may be exceeded. We formulate

Corollary 1. If in the setting of Proposition 1 the upper bound in (1) holds for all $y \in \mathbb{R}$ but the lower bound only for all $y \in [y_0, \infty)$, then a is still well-defined, increasing and right-continuous on (U^-, U^+). The parts (i)–(iii) and the U^+ statement in (iv) hold, but the U^- statement in (iv) may be violated.

If moreover $Y^- > -\infty$, we have more precisely: if there exists a $y_1 > Y^-$ such that

$$\frac{1}{y_1 - Y^-} \int_{Y^-}^{y_1} u(y)\, dy < U^- \tag{2}$$

then there is an infinite cluster in the limit made up from all particles started from $(-\infty, y_2)$ where y_2 is the supremum of all y_1 that satisfy (2). The limit speed of this cluster is U^-. There may be more clusters or regular particles with this limit speed. They can be described as in (iii) where $s=a(U^-)-y_2$.

The proofs are based on the following lemma which contains the relevant convergence results of the involved quantities. We shall need some more notation.

From the above description it is clear that $a_t(x)$ is the initial position of the left-most particle to be found to the right of x at time t. Therefore,

$$x_t(a) = \inf\{x \in \mathbb{R} : a_t(x) > a\}, \qquad a \in \mathbb{R}$$

describes the position at time t of the particle started from a and is called the Lagrangian function. We also introduce the velocity at time t of a particle started from a

$$u_t(a) = \frac{2x_t(a) - a_t(x_t(a)-) - a_t(x_t(a))}{2t}$$

Lemma 1. In the situation of Proposition 1 or Corollary 1 we have as t tends to infinity

$$a_t(tx) \to a(x) \qquad \text{for} \quad x \in (U^-, u(0)) \qquad \text{and}$$

$$a_t(tx-) \to a(x-) \qquad \text{for} \quad x \in (u(0), U^+)$$

and

$$u_t(a) = U^+ \qquad \text{for} \quad a \in [Y^+, \infty)$$

$$u_t(a) \to u(a) := \inf\{x \in \mathbb{R} : a(x) > a\} \qquad \text{for} \quad a \in (0, Y^+)$$

$$u_t(a-) \to u(a-) \qquad \text{for} \quad a \in [Y^-, 0)$$

$$(u_t(a) = U^- \qquad \text{for} \quad a \in (-\infty, Y^-))$$

i.e., we can refer to $u(a)$ as the limit velocity of the particle started in $a \in \mathbb{R}$. The statement in parentheses is valid in the setting of Proposition 1 but may fail in the setting of Corollary 1.

Proof. Let first $x \in (u(0), U^+)$ and $\varepsilon > 0$ such that $x - \varepsilon > u(0)$. It follows from the definition that a is right-continuous increasing. Due to (1) or its weaker analogue in Corollary 1, there is an $\eta > 0$ such that for all $a < a(x - \varepsilon -)$ we have

$$\int_a^{a(x-)} (u_0(y) - x)\, dy < -\eta$$

and for t big enough, say $y \geqslant t_0 = t_0(x, \varepsilon, \eta)$,

$$\int_0^a (u_0(y) - x)\, dy + \frac{a^2}{2t} - \int_0^{a(x-)} (u_0(y) - x)\, dy - \frac{(a(x-))^2}{2t} > \eta - \frac{\eta}{2}$$

$$= \frac{\eta}{2}$$

so that

$$a_t(tx-) = \arg^{-} \min_{a \in \mathbb{R}} \left\{ \int_0^a (u_0(y)-x)\, dy + \frac{a^2}{2t} \right\} > a(x-\varepsilon)$$

where $\arg^{-}$ min picks the left-most minimizing value, eventually increases in t since $a^2/2t_1 - a^2/2t_2$ is increasing in a for $t_1 < t_2$. We now show that it increases to $a(x-)$. For $t \geqslant t_0$ we have $a(x-) \geqslant a_t(tx-)$ since for all $a(x-\varepsilon-) \leqslant b < a_t(tx-)$

$$\begin{aligned} \int_0^b (u_0(y)-x)\, dy &> \int_0^{a_t(tx-)} (u_0(y)-x)\, dy + \frac{(a_t(tx-))^2}{2t} - \frac{b^2}{2t} \\ &> \int_0^{a_t(tx-)} (u_0(y)-x)\, dy \\ &\geqslant \int_0^{a(x-)} (u_0(y)-x)\, dy \end{aligned}$$

and $a(x-) \leqslant \sup_{t \geqslant t_0} a_t(tx-)$ since for all $b > \sup_{t \geqslant t_0} a_t(tx-)$ with

$$\int_0^{\sup_{t>0} a_t(tx-)} (u_0(y)-x)\, dy - \int_0^b (u_0(y)-x)\, dy > \delta > 0$$

we have, first the existence of $s \geqslant t_0$ such that

$$\frac{b^2}{2s} - \frac{(a_s(sx-))^2}{2s} < \frac{\delta}{2}$$

and then

$$\begin{aligned} 0 &\geqslant \int_0^{a_s(sx-)} (u_0(y)-x)\, dx + \frac{(a_s(sx-))^2}{2s} - \int_0^b (u_0(y)-x)\, dy - \frac{b^2}{2s} \\ &\geqslant \delta - \frac{\delta}{2} > 0 \end{aligned}$$

the required contradiction.

The corresponding convergence for $x \in (U^-, u(0))$ can be shown easily by adapting the argument. In this case $a_t(tx)$ eventually decreases to $a(x)$.

For the third and fourth convergence statements look at

$$\frac{x_t(a)}{t} = \inf\{y \in \mathbb{R} : a_t(ty) > a\} \to \inf\{y \in \mathbb{R} : a(y) > a\} = u(a)$$

for $a > 0$ and

$$\frac{x_t(a-)}{t} = \inf\{y \in \mathbb{R} : a_t(ty) > a\} \to \inf\{y \in \mathbb{R} : a(y) > a\} = u(a-)$$

for $a < 0$ which follows from the monotonicity in t and x and the convergence of $a_t(tx)$ as t tends to infinity. Now we conclude

$$u_t(a) = \frac{2x_t(a) - a_t(x_t(a)-) - a_t(x_t(a))}{2t} \to u(a)$$

for $a > 0$ and

$$u_t(a-) = \frac{2x_t(a-) - a_t(x_t(a-)-) - a_t(x_t(a-))}{2t} \to u(a-)$$

for $a < 0$ where the arguments of the a_t terms lie eventually below $2u(a)\,t$ and $a_t(x_t(a))$ is therefore bounded above.

The velocities on $(-\infty, Y^-)$ and $[Y^+, \infty)$ follow from the definitions by an elementary calculation. ∎

Proof of Proposition 1. (i) Take an interval $I_g = [g, d)$ where a is constant. Then u jumps from g to d hence, by Lemma 1, no particle has a limit speed in $[g, d)$.

(ii) Take a continuity point x of a which is not in one of the I_g. Then, due to the monotonicity of a there is a unique $a \in \mathbb{R}$ such that $u(a) = x$ which means that exactly one infinitesimal particle has limit velocity x. This particle cannot have participated in any shocks and is therefore regular.

(iii) Take a jump location x of a, say of height $s = \Delta a(x)$. Then u is constant on the corresponding interval $[a(x-), a(x))$ of length s. Obviously, there are no particles from outside this interval involved in the limit clusters and particles of limit speed x so consider only these particles surrounded by void space. Let us look at the initial speeds $u_0(a)$ of these particles a or rather $v_0(a) = u_0(a) - x$ by a centering transformation that does not change the shock evolution. Since the minimum defining $a(x)$ is attained in $a(x-)$ and $a(x)$ we have

$$\int_0^a v_0(y)\,dy \geqslant 0 \qquad \text{for all} \quad a \in [a(x-), a(x)]$$

and the minimum is attained wherever this integral vanishes. If this happens on a whole interval I, then velocities are zero inside. In fact, these particles are regular, since on every neighbouring interval the average velocity pulls away from I such that they are never involved into shocks. Similarly, for two neighbouring zeros g and d, particles from outside $[g, d]$ never enter the interval and particles from inside (g, d) never exit. Inside, the average speed at the left is strictly positive, at the right strictly negative. It is easily seen that, in the limit, all particles from (g, d) form one single clump.

(iv) This follows from the corresponding statement in Lemma 1. ▮

Proof of Corollary 1. Clearly, the above proof of Proposition 1 is still valid for the corresponding statements in Corollary 1. Also, the additional statements concerning particles started from $(-\infty, Y^-)$ follow using the same arguments. ▮

3. STRUCTURE OF LIMIT CLUSTERS

3.1. The Markov Process Case

We shall only consider "nice" Markov processes, i.e., time homogeneous strong Markov processes that admit a càdlàg version. We work with a standard model, i.e., the underlying probability space $(\Omega, \mathscr{A}, (P_x)_{x \in \mathbb{R}})$ is the Skorokhod space of càdlàg paths $\mathscr{D}(\mathbb{R}_+, \mathbb{R})$ or $\mathscr{D}(\mathbb{R}, \mathbb{R})$ equipped with probability measures P_x, $x \in \mathbb{R}$ such that $X_t(\omega) = \omega(t)$ is the Markov process starting from x under P_x. We may then also use the shift and killing operators defined respectively by

$$(\theta_t(\omega))(s) = \omega(t+s), \quad (k_t(\omega))(s) = \omega(s)\, 1_{\{s \leqslant t\}}$$

and the natural filtration of X by $\mathscr{F}_t = \sigma(X_s, s \leqslant t)$ suitably extended to be right-continuous and complete. The Markov property can then be expressed as follows: for every a.s. finite $\mathscr{F}$-stopping time τ we have

$\theta_\tau X$ and $k_\tau X$ are conditionally independent given X_τ.

The law of $\theta_\tau X$ under P_x is P_{X_τ}

We can define the associated process of jumps

$$\Delta X_s = X_s - X_{s-}, \qquad s \geqslant 0$$

We say that X is spectrally negative if the whole process ΔX does not take positive values a.s. This restriction still leaves a large class of processes, e.g., Markov processes with infinitesimal generator an extension of

$$(\mathscr{A}f)(x) = \gamma(x)\, f'(x) + \tfrac{1}{2}\sigma^2(x)\, f''(x) + \int_{\mathbb{R}} (f(x+y) - f(x) + 1_{\{|y|<1\}}\, y f'(x))\, \Pi(x, dy) \tag{3}$$

are spectrally negative if $\Pi(x, (0, \infty)) = 0$ for all $x \in \mathbb{R}$. Such processes exist, if γ, σ^2 and Π are sufficiently regular in x, furthermore $1 \wedge y^2$ integrable w.r.t. $\Pi(x, dy)$. We refer e.g., to Theorem (VII.1.13) of Revuz and Yor[17] or to Jacob and Schilling.[18]

Our result is on the limit a of the inverse Lagrangian functions a_t in terms of which we described the limit clusters in Proposition 1:

Theorem 1. Let X be a càdlàg strong Markov process with no positive jumps such that $X_y \to \infty$ a.s. as y tends to infinity. Then

$$a(x) = \arg^+ \min_{a \geqslant 0} \left\{ \int_0^a (X_y - x)\, dy \right\}$$

is increasing and right-continuous. Under P_{x_0}, $a(x) - a(x_0)$, $x \geqslant x_0$, is a process with independent increments, and independent of $a(x_0)$.

Proof. We adapt the proof of Theorem 1 of Bertoin.[10] First note that $a(x)$ is positive and finite P_{x_0}-a.s. for all $x \geqslant x_0$. We refer to Proposition 1 for further properties of a. Fix $x \geqslant x_0$ and define processes

$$I_y^x = \int_0^y (X_z - x)\, dz, \qquad m_y^x = \inf_{0 \leqslant z \leqslant y} \{I_z^x\} \qquad \text{and} \qquad D_y^x = I_y^x - m_y^x$$

We have for any stopping time T

$$D_{T+y}^x = \begin{cases} D_T^x + \theta_T I_y^x & \text{if} \quad D_T^x + \theta_T I_\eta^x \geqslant 0 \qquad \text{for all} \quad 0 \leqslant \eta \leqslant y \\ \theta_T I_T^x - \inf_{0 \leqslant y \leqslant T} \{\theta_y I_y^x\} & \text{otherwise} \end{cases}$$

This shows the strong Markov property for the bivariate process (X, D^x) since we can deduce that (X_{T+y}, D_{T+y}^x) depends on $\mathscr{F}_T$ only via X_T and D_T^x. Furthermore, D^x is continuous, so (X, D^x) is càdlàg. Now

$$a(x) = \sup\{y > 0 : D_y^x = 0\}$$

implies $X_{a(x)} = x$ (by the absence of positive jumps for X, see also the definition of $a(x)$ as an integral of $X_y - x$) and is therefore the last exit time from $(0, x)$. By Theorem (2.12) of Getoor[19] splitting at last exit times yields two independent processes

$$k_{a(x)}(X, D^x) \qquad \text{and} \qquad \theta_{a(x)}(X, D^x)$$

Due to the monotonicity of a, the process $a(z)$, $x_0 \leqslant z \leqslant x$, is a functional of $k_{a(x)}(X, D^x)$ whereas for any $y \geqslant 0$

$$\begin{aligned} a(x+y) &= \arg^+ \min_{a \geqslant 0} \{I_a^{x+y}\} \\ &= \arg^+ \min_{a \geqslant a(x)} \{I_{a(x)}^{x+y} + \theta_{a(x)} I_{a-a(x)}^{y}\} \\ &= a(x) + \arg^+ \min_{a \geqslant 0} \{\theta_{a(x)} I_a^y\} \end{aligned}$$

where we used $X_{a(x)} = x$ and the independence from a of the first summand in $\arg^+ \min$. This completes the proof. ∎

The assumption that X tends to infinity is essential for the validity of the theorem. Typically, even a zero expectation leads to infinite limit clusters. For instance, if X is Brownian motion, one deduces from Theorem 1 in ref. 10 by a simple scaling argument that, as t tends to infinity

$$a_t(0) \sim t^2 a_1(0) \to \infty$$

3.2. The Lévy Process Case

A Lévy process is a space-time homogeneous Markov process. It can be characterized by its generator on $\mathscr{C}_b^2$. In fact, X is a Lévy process if and only if in (3) the characteristics γ, σ^2 and Π do not depend on x. This means in particular, that the laws P_x are simply translations of each other and the Markov property can be stated as: for every a.s. finite $\mathscr{F}$-stopping time τ

$$\theta_\tau X - X_\tau \text{ and } k_\tau X \text{ are independent.}$$

$$\text{The law of } \theta_\tau X - X_\tau \text{ under } P_x \text{ is } P_0$$

The characteristics γ and σ^2 describe the continuous component of a Lévy process X which is $\gamma t + \sigma W_t$ for a Brownian motion W. ΔX is a homogeneous Poisson point process with intensity measure Π. In particular, X has a discrete jump structure if and only if Π is a finite measure. A Lévy process has bounded variation if and only if $\sigma^2 = 0$ and $1 \wedge |y|$ is integrable w.r.t. Π (jumps summable without compensation). We then introduce

$$\gamma' = \gamma + \int_{\mathbb{R}} 1_{\{|y| \leqslant 1\}} \, y \Pi(dy)$$

and see that X_t is a linear drift $\gamma' t$ plus jumps according to the Poisson point process with intensity measure Π which is not the case in general since the integral defining γ' does not converge and is indeed needed as a compensation term in the integral of (3) (jumps are not summable, in general). A Lévy process is called a subordinator if its paths are a.s. increasing. This is equivalent to having bounded variation, $\gamma' \geqslant 0$ and $\Pi((-\infty, 0)) = 0$.

As standard references on Lévy processes we mention Bertoin[20] and Sato,[21] also Bertoin[22] for more details on subordinators.

We can identify the law of the process

$$a(x) = \arg^+ \min_{a \geqslant 0} \left\{ \int_0^a (X_y - x) \, dy \right\}$$

in this case:

Proposition 2. Let X be a Lévy process with no positive jumps such that $X_y \to \infty$ a.s. as y tends to infinity. Then

$$(a(x) - a(0))_{x \geqslant 0} \sim (T(x))_{x \geqslant 0}$$

where $T(x)$, $x \geqslant 0$, is the first passage time subordinator of X.

Proof. One may either conclude from Theorem 2 of Bertoin[10] by passing to the limit $t \to \infty$ or adapt the proof of our Theorem 2 below to this simpler situation. ∎

3.3. The Self-Similar Case

Let Y be a Lévy process with $Y_s \to \infty$, $s \to \infty$ and

$$X_z = \exp\{Y_{\rho_z}\} \qquad \text{where} \quad \rho_z = \inf\left\{ s \geqslant 0 : \int_0^s \exp\{\alpha Y_u\} \, du > z \right\} \tag{4}$$

the associated self-similar (sometimes also called semi-stable) process of order $\alpha \geqslant 0$ on $(0, \infty)$, cf. e.g., Lamperti[23] where he called $1/\alpha$ the index whereas we also allow the order $\alpha = 0$: the time change is then the trivial $\rho_z = z$, and X is just the exponential process $X = \exp\{Y\}$. For all $\alpha \geqslant 0$, X is a strong Markov process such that $X_t \to \infty$, $t \to \infty$. If Y is spectrally negative, then so is X. If $Y_0 = \log(x)$ then $X_0 = x$. Denote by X^x the process X constructed from $Y + \log(x)$ for all $x > 0$. Then X has the following scaling property:

$$(kX^x_{k^{-\alpha}z})_{z \geqslant 0} \sim (X^{kx}_z)_{z \geqslant 0}$$

Lamperti showed, that all self-similar càdlàg strong Markov processes on $(0, \infty)$ are of the form (4) for a Lévy process Y that can be recovered by

$$Y_u = \log(X_{C_u}) \qquad \text{where} \quad C_u = \inf\left\{ t \geqslant 0 : \int_0^t X_s^{-\alpha} ds > u \right\}$$

It follows immediately from the definition that the first passage time process T^X of X^1 can be given in terms of Y and its first passage time process T^Y as follows

$$T^X_x = \int_0^{T^Y_{\log(x)}} \exp\{\alpha Y_u\}\, du, \quad x \geqslant 1$$

Define the processes

$$a(x) = \arg\min_{a \geqslant 0} \left\{ \int_0^a (X^1_z - x)\, dz \right\}, \qquad x \geqslant 1 \qquad \text{and}$$

$$b(x) = a(e^x), \quad x \geqslant 0$$

Then our result is

Theorem 2. When X is a spectrally negative self-similar Markov process of order $\alpha \geqslant 0$, then

$$a(x) - a(1) \sim T^X_x = \int_0^{\log(x)} e^{\alpha y}\, dT^Y_y$$

where Y is the associated Lévy process. This holds as an identity in law of processes, $x \geqslant 1$.

Proof. We start by a calculation similar to the one in the proof of Theorem 1 using the Markov property but also the scaling property

$$
\begin{aligned}
b(x+y) &= \arg\min_{a \geqslant T^X_{e^x}} \left\{ \int_{T^X_{e^x}}^{a} (X^1_z - e^{x+y})\, dz \right\} \\
&= T^X_{e^x} + \arg\min_{b \geqslant 0} \left\{ \int_0^b (X^1_{T^X_{e^x}+z} - e^{x+y})\, dz \right\} \\
&\sim T^X_{e^x} + \arg\min_{b \geqslant 0} \left\{ \int_0^b (\tilde{X}^{e^x}_z - e^{x+y})\, dz \right\} \\
&\sim T^X_{e^x} + \arg\min_{b \geqslant 0} \left\{ \int_0^b (e^x \tilde{X}^1_{e^{-\alpha x}z} - e^{x+y})\, dz \right\} \\
&= T^X_{e^x} + \arg\min_{b \geqslant 0} \left\{ e^{x(1+\alpha)} \int_0^{e^{-\alpha x}b} (\tilde{X}^1_\zeta - e^y)\, d\zeta \right\} \\
&= T^X_{e^x} + \arg\min_{b \geqslant 0} \left\{ \int_0^{e^{-\alpha x}b} (\tilde{X}^1_\zeta - e^y)\, d\zeta \right\} \\
&= T^X_{e^x} + e^{\alpha x} \arg\min_{c \geqslant 0} \left\{ \int_0^c (\tilde{X}^1_\zeta - e^y)\, d\zeta \right\} \\
&= \int_0^{T^Y_x} \exp\{\alpha Y_u\}\, du + e^{\alpha x}\tilde{b}(y)
\end{aligned}
$$

and this is an identity in law of processes, $y \geqslant 0$. We deduce in particular that

$$
(b(x+y) - b(x))_{y \geqslant 0} \sim e^{\alpha x}(b(y) - b(0))_{y \geqslant 0} \tag{5}
$$

for all $x \geqslant 0$. By Theorem 1 b is a process with independent increments. We define another process with independent increments by

$$
\sigma_x = \int_{[0-, z]} e^{-\alpha z}\, db(z) = b(0) + \int_{(0, z]} e^{-\alpha z}\, db(z)
$$

and note that due to (5)

$$
\sigma_{x+y} - \sigma_x = \int_{(0, y]} e^{-\alpha(x+z)}\, db(x+z) \sim \int_{(0, y]} e^{-\alpha z}\, db(z) = \sigma_y - \sigma_0
$$

i.e., σ is a subordinator.

We now show that σ and T^Y have the same increment distribution, i.e., $d\sigma \sim dT^Y$ on $(0, \infty)$. In fact our argument is a comparison of stationary limit laws of Ornstein–Uhlenbeck type processes (OU processes). We define the OU process B associated to σ as

$$B_x = e^{-\alpha x} \int_{[0-,x]} e^{\alpha y}\, d\sigma_y = e^{-\alpha x}\sigma(0) + e^{-\alpha x} \int_{(0,x]} e^{\alpha y}\, d\sigma_y = e^{-\alpha x} b(x)$$

and repeat the same procedure to define

$$T_x = e^{-\alpha x} T^X_{e^x} = e^{-\alpha x} \int_0^{T^Y_x} \exp\{\alpha Y_u\}\, du = e^{-\alpha x} \int_{[0,x]} e^{\alpha y}\, dT^Y_y$$

via the substitution $u = T^Y_y$ and similarly

$$D_x = e^{-\alpha x} D^X_{e^x} = e^{-\alpha x} \int_{[0-,x]} e^{\alpha y}\, dD^Y_y = e^{-\alpha x} D^Y_0 + e^{-\alpha x} \int_{(0,x]} e^{\alpha y}\, dD^Y_y$$

where D^Y and D^X are the last passage time processes associated with Y and X^1, respectively. By a standard time reversal argument for spectrally negative Lévy processes (cf. Theorem VII.18 in ref. 20 for a related result), we have $dD^Y \sim dT^Y$ on $(0, \infty)$. Furthermore, since $b(x) = a(e^x)$ as well, is a passage time of e^x by X^1 (due to the absence of positive jumps, as noted in the proof of Theorem 1), we have $T_x \leqslant B_x \leqslant D_x$. Clearly, $Y_y \to \infty,\ y \to \infty$, ensures

$$E(T_1) < \infty \Rightarrow E((\log T_1)^+) < \infty$$

so by Theorem 17.5.(i) in Sato,[21] T and D have weak limit distributions which coincide due to $dT^Y \sim dD^Y$ on $(0, \infty)$. It is now easily seen (e.g., using Laplace transforms) that also the law of B_x tends to the same limit. By Theorem 17.5.(ii) in ref. 21 we conclude that $d\sigma \sim dT^Y$ on $(0, \infty)$. ∎

Note that the case $\alpha = 0$ of Theorem 2 is particularly simple:

Corollary 2. If $X = \exp\{Y\}$ for a spectrally negative Lévy process Y, then

$$(a(x) - a(1))_{x \geqslant 1} \sim (T(\log(x)))_{x \geqslant 1}$$

where $T(y)$, $y \geqslant 0$, is the first passage time subordinator of Y. If $T(1)$ has characteristics $(b, 0, \Pi)$, then $a(x+y) - a(x)$ has characteristics $(\log(x+y) - \log(x))(b, 0, \Pi)$.

Theorem 1 applies to $X = f(Y)$ for all strictly increasing continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, whereas $f = \exp$ is essential for the validity of Corollary 2. More precisely, for general f, $a(f(x)) - T_x$ is independent of $k_{T(x)}X$, but the law of

$$a(f(x)) - T_x \sim \arg^+ \min_{a \geqslant 0} \left\{ \int_0^a (f(x+Y_y) - f(x)) \right\} \tag{6}$$

depends on x, in general. E.g., take f linear apart from an increase of slope in x_1 and a decrease of slope in x_2 large and far away from each other. Then for $x = x_1$, the integrals over the initial positive parts of excursions get more weight than the integrals over the final negative parts of excursions, such that the minimum in (6) is typically found close to the origin. For $x = x_2$, the opposite is the case, such that the minimum is closer to the maximal possible value $D_0 = \sup\{a \geqslant 0 : Y_a = 0\}$. A time reversal argument (as applied by Bertoin[10]) shows that for $f = id$ the minimum position has a symmetric distribution on $[0, D_0]$, whereas for $f = \exp$, the distribution is biased towards the origin. For a less particular f, this bias or at least the law depend on x.

3.4. Consequences of the Main Results

In Proposition 1 we represented the structure of limit clusters in terms of a. In Theorem 2 we described the law of a in terms of a subordinator. This allows to deduce several corollaries that we only state for X self-similar or the limit case $X = \exp\{Y\}$. Of course, there is always an obvious analogue for $X = Y$.

Corollary 3. Let X be a self-similar process associated to a spectrally negative Lévy process Y such that $Y_t \to \infty$ as $t \to \infty$. Then the set of regular particles is given by

$$\mathcal{R} = \{a(x) : x \geqslant u(0),\, a \text{ continuous in } x\}$$

The limit shock structure is as follows.

(i) The shock structure is discrete if and only if the jump structure of Y is discrete. Then $\mathcal{R}$ is a countable union of intervals.

(ii) $\mathcal{R}$ has positive Lebesgue measure if and only if Y has bounded variation. In general, its Hausdorff dimension is given by

$$\dim(\mathcal{R}) = \sup\{\alpha > 0 : \lim_{\lambda \to \infty} \lambda^{-1/\alpha}\Psi(\lambda) = 0\} \in [1/2, 1]$$

where Ψ is the Laplace exponent of Y.

Proof. First note that $Y_t \to \infty$ for a spectrally negative Lévy process is only possible if Y has unbounded variation or a positive drift coefficient. This implies that a is strictly increasing since T^Y is, cf. ref. 20 section VII.1. Therefore the results follow from Proposition 1 and Theorem III.15 in ref. 20 which gives the dimension of the range of a subordinator. ∎

Corollary 4. Let X be a self-similar Markov process associated to a spectrally negative Lévy process Y such that $Y_t \to \infty$ as $t \to \infty$. Denote by

$$s(u) = \mathcal{M}(\{a \geqslant 0 : \lim_{t \to \infty} u_t(a) = u\}), \qquad u \geqslant 1$$

the total size of limit clusters with limit speed u. Then $\{(u, s(u)) : u \geqslant 1, s(u) > 0\}$ is a Poisson cloud in $[1, \infty) \times (0, \infty)$ with intensity measure $\Pi(z^{-\alpha} d\eta)\, z^{-1}\, dz$.

Proof. The function $s(u)$, $u \geqslant 1$ specifies the jump sizes of a. From (5) we deduce that the Poisson cloud of jumps of b has the intensity measure $\Pi(e^{-\alpha x}\, d\eta)\, dx$ on $[0, \infty) \times (0, \infty)$. An elementary transformation completes the proof. ∎

The Poisson property is actually also true in the general situation of Theorem 1. The intensity measure is given by $v([0, u] \times ds) = \Pi_u(ds)$ where Π_u is the Lévy measure of $a(u)$. However, this is just another way of formulating the statement of this theorem. The point of the corollary is the explicit nature of the intensity measure.

Corollary 5. Let $X = \exp\{Y\}$ for a spectrally negative Lévy process Y, such that $Y_t \to \infty$ as $t \to \infty$. Denote by $\ell(y)$ the limit cluster size of the particle started from y. Then

$$P(\ell(y + a(1)) \in dz) = bv(y)\, \delta_0 + (V(y) - V(y - z))\, \Pi(dz)$$

where $V(x) = E(S_x)$ has a continuous derivative v on $(0, \infty)$ if the drift coefficient b of T is positive, $S_x = \sup_{0 \leqslant y \leqslant x} Y_y$, $x \geqslant 0$, is the renewal measure of the first passage time process T and Π is the Lévy measure of T.

The condition $Y_t \to \infty$ ensures $\mu := E(T_1) < \infty$. The laws of $\ell(y)$ converge weakly as y tends to infinity to the limit law $b\mu^{-1}\delta_0(dz) + \mu^{-1} z \Pi(dz)$.

To prove this corollary, there is a technicality to be ruled out being the content of

Lemma 2. Let X be a self-similar Markov process associated to a spectrally negative Lévy process Y satisfying $Y_y \to \infty$, $y \to \infty$, (or $X = Y$). Then the following holds a.s. For all $x > 0$ (or $x \in \mathbb{R}$, respectively)

$$I_y^x = \int_0^y (X_z - x)\, dz$$

attains its overall minimum at most twice.

Proof. This proof requires several properties of Lévy processes for which we refer to Bertoin.[20]

Note first, that a.s. for all $x > 0$ (or $x \in \mathbb{R}$) I^x takes all its local minima on $\{y \geqslant 0 : X_y = x\}$ since X is spectrally negative. We can therefore restrict our attention to these points.

If Y has bounded variation, the hitting times of x form a discrete set, more precisely, by the Markov property, they are described by an increasing random walk T_n^x, $n \geqslant 1$, with a geometric life-time. We also associate the random walk $J_n^x = I^x(T_n^x)$ which is easily seen to have a non-atomic step distribution F.

Define for all $a \in \mathbb{Q} \cap [0, \infty)$ a stopping time

$$T(a) = \inf\{z > a : \inf_{0 \leqslant y \leqslant a} I_y^{X_z} = I_z^{X_z} = \inf_{0 \leqslant y \leqslant z} I_y^{X_z}\}$$

$\{T(a) : a \in \mathbb{Q} \cap [0, \infty)\}$ contains all those positions at which an I^x attains its pre-minimum for a second time as a local minimum. The local minimum property is due to the fact that X does not immediately decrease after stopping times. This property is called irregularity of 0 for $(-\infty, 0]$ for Y. By the Markov property of X, the random time

$$N(a) = \inf\{n \geqslant 0 : T_n^{T(a)} = T(a)\}$$

is such that $(J_n^{X(T(a))})_{n \leqslant N(a)}$ is measurable w.r.t. $\mathscr{F}_{T(a)}$, and $(J_{N(a)+n}^{X(T(a))} - J_{N(a)}^{X(T(a))})_{n \geqslant 0}$ is independent of $\mathscr{F}_{T(a)}$ and distributed as J^1 (or J^0). In particular, it does not hit zero a.s. by the non-atomicity of F. Therefore, the local minimum is not attained for a third time.

If Y has unbounded variation, the hitting times of x do not form a discrete set but the concept of local times allows to adapt this argument. We leave the details to the interested reader. He may need to ensure the local minimum property by postulating that the minimal value is attained in $T(a, \varepsilon) - \varepsilon$ and is not exceeded within distance of a rational $\varepsilon > 0$. ∎

Proof of Corollary 5. By Lemma 2 there is at most one limit cluster (between the two minima of I^x) associated with any limit velocity x.

Therefore $\ell(y) = s(u(y))$ in the notation of the preceding corollary. Now define $L_y = \inf\{z \geqslant 0 : T_z^Y \geqslant y\}$ and calculate

$$\begin{aligned} u(y+a(1)) &= \inf\{x \geqslant 1 : a(x) - a(1) \geqslant y\} \\ &= \exp\{\inf\{z \geqslant 0 : a(e^z) - a(1) \geqslant y\}\} \\ \ell(y+a(1)) &= s(u(y+a(1))) \\ &\sim \Delta T^Y(L_y) \end{aligned}$$

The laws of these quantities and their limit behaviour are well-known, cf., e.g., ref. 22, Lemma 1.10 and Proposition 1.6 (Renewal Theorem).

The limit law of $\ell(y)$ is the same as that of $\ell(y+a(1))$ since, by Theorem 1, $\ell(y+a(1))$ is independent of $a(1)$, and therefore by dominated convergence

$$\begin{aligned} Ef(\ell(y)) &= \int_{[0,\infty)} f(\ell(y-x+a(1)))\, P(a(1) \in dx) \\ &\to \int_{[0,\infty)} \left(f(0)\, b\mu^{-1} + \int_{(0,\infty)} f(z)\, \mu^{-1} z \Pi(dz) \right) P(a(1) \in dx) \\ &= \int_{[0,\infty)} f(z)\, \mu^{-1}(b\delta_0(dz) + z\Pi(dz)) \end{aligned}$$

for every bounded continuous real function f. ∎

Look again at the proof of Lemma 2, when the jump structure of Y is discrete (and the drift positive), say. Note that even in this simple case, the touching straight lines when approximated by parabola segments do not attain their limit touching points at a finite time. This means that no limit cluster is established at a finite time. When the jump structure is not discrete, note that the touching points are not jump times of Y since I^x would otherwise change its slope in the wrong direction. So I^x is continuously differentiable in the touching points (but not in a whole neighbourhood). This means that approximating parabolas have strictly different touching points from their limiting straight line.

Before passing to the two-sided case, we briefly look at the initial value $a(1)$ which gives the initial position of the left-most particle that has a limit speed greater than 1. We can deduce from the regularity properties of Y that $P(a(1)=0)=0$ if Y has unbounded variation and $0 < P(a(1)=0) < 1$ if Y has bounded variation ($Y_y = \gamma y$ being a trivial exception).

3.5. The Two-Sided Case

We say that X is a doubly infinite Lévy process if

$$X_y = Y_y, \qquad X_{-y} = -\tilde{Y}_{y-}, \qquad y \geqslant 0$$

for two independent identically distributed Lévy processes Y and $\tilde{Y}$ (starting from zero).

Theorem 3. (i) When X is a doubly infinite Lévy process with no positive jumps such that $X_y \to \infty$ as $y \to \infty$, then

$$(a(x) - a(0))_{x \in \mathbb{R}} \sim (T(x) - T(0))_{x \in \mathbb{R}}$$

where

$$T(x) = \inf\{y \in \mathbb{R} : X_y > x\}, \qquad x \in \mathbb{R}$$

is the doubly infinite stationary first passage time subordinator of X.

(ii) When $X = \exp\{Z\}$ for a spectrally negative doubly infinite Lévy process Z, then

$$(a(x) - a(1))_{x>0} \sim (T(\log(x)) - T(0))_{x>0}$$

where $T(y)$, $y \in \mathbb{R}$, is the first passage time subordinator of Z. If T has characteristics $(b, 0, \Pi)$, then $a(x+y) - a(x)$ has characteristics $(\log(x+y) - \log(x))(b, 0, \Pi)$.

Before proving the theorem, we present the approximation method which allows to conclude from Theorem 2. The following lemma concerns the first passage time subordinator that we need to know for the proof of the theorem as well. We need some terminology and preliminaries.

Closed ranges $\mathscr{R} = \{X_t : t \geqslant 0\}^{cl}$ of subordinators X are called regenerative sets. If $\mathscr{R}^+$ and $\mathscr{R}^-$ are the ranges of two independent identically distributed subordinators X and Y with $\mu = E(X_1) < \infty$, then there is a law $P(g_0 \in \cdot, d_0 \in \cdot)$ on $(-\infty, 0] \times [0, \infty)$ such that for (g_0, d_0) independent of X and Y the set $\mathscr{R} = (d_0 + \mathscr{R}^+) \cup (g_0 - \mathscr{R}^-)$ is stationary in the sense that $\mathscr{R} - t \sim \mathscr{R}$ for all $t \in \mathbb{R}$. The law of (g_0, d_0) can be given in terms of the characteristics of X as

$$P(d_0 - g_0 \in dz, g_0 \in dy) = \frac{b}{\mu} \delta_{(0,0)}(dz \times dy) + \frac{1}{\mu} 1_{\{z \geqslant -y \geqslant 0\}} \, dy \, \Pi(dz) \qquad (7)$$

We refer to Fristedt[24] for a survey on regenerative sets. See also Bertoin.[22]

Lemma 3. Let $Y = (Y_y)_{y \geqslant 0}$ be a Lévy process drifting to $+\infty$. Then

$$(Z_y^n = Y_{(y+n) \vee 0} - Y_n)_{y \in \mathbb{R}} \xrightarrow[n \to \infty]{(d)} (Z_y)_{y \in \mathbb{R}}$$

where Z is the corresponding doubly infinite Lévy process. Here (d) denotes functional convergence in the Skorokhod sense.

Furthermore, their first passage time processes

$$T_x^n = \inf\{y \in \mathbb{R} : Z_y^n > x\}, \qquad x \in \mathbb{R}$$

converge as well in the Skorokhod sense. In the spectrally negative case, the limit process T is a stationary two-sided subordinator in the sense that its closed range $\mathscr{R} = \{T_x : x \in \mathbb{R}\}^{cl}$ is a stationary regenerative set.

Proof. We shall consider a copy $\tilde{Z}^n$ of Z^n coupled to Z and a Lévy process $\bar{Z}^n$ as follows:

$$\tilde{Z}_y^n = Z_{y \vee (-n)} \qquad \text{and} \qquad \bar{Z}_y^n = Z_y^n + Y_n = Y_{(y+n) \vee 0}$$

The coupling copy $\tilde{Z}^n$ will provide us with convergence results. The Lévy process $\bar{Z}^n$ will allow us to analyse the limits.

Obviously the coupling ensures

$$\tilde{Z}^n \xrightarrow[n \to \infty]{a.s.} Z \quad \Rightarrow \quad Z^n \xrightarrow[n \to \infty]{(d)} Z \quad \text{locally uniformly}$$

Also, using $Z_{-y} \to -\infty$, $y \to \infty$, we deduce that in the obvious notation

$$\tilde{T}^n \xrightarrow[n \to \infty]{a.s.} T \quad \Rightarrow \quad T^n \xrightarrow[n \to \infty]{(d)} T \quad \text{locally uniformly}$$

In the spectrally negative case, the subordinator property is well known for $(T_x^0)_{x \geqslant 0}$, cf. Theorem VII.1 in ref. 20. Denoting $\mathscr{R}^0 = \{T_x^0 : x \geqslant 0\}^{cl}$ the closed range of T^0, it is also well known that the process of overshoots and undershoots $M_y^0 = (d_y^0 - y, y - g_y^0)$, where $d_y^0 = \inf\{z \geqslant y : z \in \mathscr{R}^0\}$ and $g_y^0 = \sup\{z \leqslant y : z \in \mathscr{R}^0\}$, approaches a stationary regime ν determined by

$$P(d - g \in dz, g \in dy) = \frac{b}{\mu}\, \delta_{(0,0)}(dz \times dy) + \frac{1}{\mu}\, 1_{\{z \geqslant -y \geqslant 0\}}\, dy\, \Pi(dz)$$

where $(b, 0, \Pi)$ are the characteristics of T^0 and $\mu = E(T_1^0)$, cf. ref. 22, Lemma 1.10 and Proposition 1.6 (Renewal Theorem). M^0 is easily seen to be a positive recurrent càdlàg strong Markov process, cf. Exercise IV.6.2(b) in ref. 20. M^0 does not change when shifting Z^0 to start from any non-zero Z_0^0. Therefore, in the obvious notation,

$$M^n = \bar{M}^n = M^0_{n+\cdot} \xrightarrow[n\to\infty]{(d)} M \quad \text{locally uniformly}$$

by the ergodic theorem for Markov processes where M is a stationary Markov process. The suitable ergodic theorem can be established by coupling arguments. In the bounded variation case $\mathscr{R}^0$ is heavy (i.e., has positive Lebesgue measure) since Z^0 drifts to ∞ and only increases by its deterministic drift component $\gamma' t$, cf. also section VII.1 of ref. 20. Therefore M^n couples with an independent stationary Markov process, cf. Bertoin,[25] and the result is, e.g., Theorem 7.4.1. of Thorisson.[26] In general, in the unbounded variation case, the independent coupling is not successful. Instead, we construct a successful coupling as follows: $\mathscr{R}^0$ is light, so the stationary law ν does not charge $(0, 0)$ and the Skorokhod embedding of ν, cf. Bertoin and Le Jan,[27] provides a stopping time T such that $M^0_T \sim \nu$. Using T as a coupling time to a stationary process, we obtain the ergodic theorem.

Note that M is not just some limit process but

$$\tilde{M}^n \xrightarrow[n\to\infty]{a.s.} M \quad \Rightarrow \quad M^n \xrightarrow[n\to\infty]{(d)} M \quad \text{locally uniformly}$$

shows that it is the correct functional of Z. In fact, whenever we wrote locally uniform convergence, we could have been more precise in that the limit is attained at an a.s. finite $n = N(x)$ uniformly on $[x, \infty)$. This type of convergence implies also that

$$\begin{aligned}\int_A^{\cdot} 1_{\tilde{\mathscr{R}}^n}(y)\, m(dy) &= \int_A^{\cdot} 1_{\{\tilde{M}^n_y = (0,0)\}} m(dy)\\ &\xrightarrow[n\to\infty]{a.s.} \int_A^{\cdot} 1_{\{M_y = (0,0)\}} m(dy) = \int_A^{\cdot} 1_{\mathscr{R}} m(dy)\end{aligned}$$

locally uniformly for every real-valued random variable A and every non-atomic σ-finite measure m. By choosing $A = T_0$ and m an exact Hausdorff measure of $\mathscr{R}^0$, cf. Fristedt and Pruitt,[28] this is the locally uniform convergence of local times on $\mathscr{R}$. By passing to the right-continuous inverse we see that T is the inverse local time of the stationary regenerative set $\mathscr{R}$. ∎

As part of the proof of Lemma 3 we obtained

Corollary 6. In the situation of Lemma 3 the process $(d_y - y, y - g_y)_{y\in\mathbb{R}}$ of overshoots and undershoots of T is stationary. Here $d_y = \inf\{z \geqslant y : z \in \mathscr{R}\}$ and $g_y = \sup\{z \leqslant y : z \in \mathscr{R}\}$. The stationary distribution (the law of (d_0, g_0)) is determined by (7) where $(b, 0, \Pi)$ are the characteristics of T and $\mu = E(T_1 - T_0) = b + \int_{(0,\infty)} x\Pi(dx)$.

Note that the law of T is determined by its range $\mathscr{R}$ only up to a linear transformation. Two more ingredients have to be specified: the law of the location

$$L_0 = \sup_{y \leqslant 0} \{Z_y\}, \quad E(\exp\{-\lambda L_0\}) = \frac{\lambda}{\mu\Phi(\lambda)}$$

of the passage of T across zero and the deterministic mean drift μ. Cf. ref. 20, VII.(3) for the law of L_0 in terms of the Laplace exponent

$$\Phi(\lambda) = -\log E \exp\{-\lambda(T_{t+1} - T_t)\} = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \Pi(dx)$$

of the stationary subordinator T.

Proof of Theorem 3 (i) We obtain this from Theorem 2 by the limiting procedure

$$(Z^n_y = Y_{(y+n)\vee 0} - Y_n)_{y \in \mathbb{R}} \xrightarrow[n\to\infty]{(d)} (Z_y)_{y \in \mathbb{R}}$$

that we analysed in Lemma 3. In this lemma we obtained the convergence

$$(T^n_x)_{x \in \mathbb{R}} \xrightarrow[n\to\infty]{(d)} (T_x)_{x \in \mathbb{R}}$$

of first passage time processes

$$T^n_x = \inf\{y \in \mathbb{R} : Z^n_y \geqslant x\}, \qquad x \in \mathbb{R}$$

where the limit T is a stationary subordinator having the transition kernel of T^0.

The very same argument as in Lemma 3 also applies to show that

$$(a^n(x))_{x \in \mathbb{R}} \xrightarrow[n\to\infty]{(d)} (a(x))_{x \in \mathbb{R}}$$

where

$$a^n(x) = \arg^+ \min_{a \in \mathbb{R}} \left\{ \int_0^a (Z^n_y - x)\, dy \right\}$$

and the limit a is a stationary subordinator having the transition kernel of a^0.

(ii) We adapt the argument in (i) which gives us

$$(a^n(\exp\{x\}))_{x\in\mathbb{R}} \xrightarrow[n\to\infty]{(d)} (a(\exp\{x\}))_{x\in\mathbb{R}}$$

where for $z>0$

$$a^n(z) = \arg^+\min_{a\in\mathbb{R}} \left\{ \int_0^a (Z_y^n - z)\, dy \right\}$$

Here we obtain $a \circ \exp$ as a stationary subordinator having the transition kernel of $a^0 \circ \exp$. ∎

Note that we do not have $a \sim T$ since $T(0)$ is supported by $(-\infty, 0]$ whereas $a(0)$ is supported by $\mathbb{R}$ and, in particular, can take positive values.

Corollary 7. Let $X = \exp\{Z\}$ for a spectrally negative doubly infinite Lévy process Z, then for all $y \in \mathbb{R}$

$$P(\ell(y) \in dz) = \beta\mu^{-1}\delta_0(dz) + \mu^{-1} z \Pi(dz)$$

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